Lecture 4

Thursday, 5 February 2026 8:41 AM

Consider the LP in std. form: | 1.5, 2.5, 0, 11

min $4x_1 + 2x_2 - 2x_3$

s.t. $2x_1 - 2x_2 + x_3 = -2$ $\times y_1$

$x_1 + x_2 + x_3 = 4$ $\times y_2$

$x_1, x_2, x_3 \geq 0$

Let OPT be optimal value, x_1^*, x_2^*, x_3^* be optimal soln.

- For any feasible x, $x_1(2y_1 + y_2) + x_2(-2y_1 + y_2) + x_3(y_1 - y_2) = -2y_1 + 4y_2$
- Say we choose y_1, y_2 so that coeffs of $x_1, x_2, x_3 \leq$ coeffs in objective:

$2y_1 + y_2 \leq 4$

$-2y_1 + y_2 \leq 2$

$y_1 - y_2 \leq -2$

then for any feasible x, the objective $-2y_1 + 4y_2 \leq 4x_1 + 2x_2 - 2x_3$. In particular, $-2y_1 + 4y_2 \leq$ OPT. This thus a lower bound on OPT.

- So maximize this lower bound:

max $-2y_1 + 4y_2$

s.t. $2y_1 + y_2 \leq 4$

$-2y_1 + y_2 \leq 2$

$y_1 - y_2 \leq -2$

and any feasible soln. to this LP is a lower bound on OPT.

This is called the dual.

For any feasible x_1, x_2, x_3, y_1, y_2: $-2y_1 + 4y_2 \leq 4x_1 + 2x_2 - 2x_3$; OPT (dual) $\leq$ OPT (primal)

Consider soln: $y_1 = 1/2$, $y_2 = 3$; $x_1 = 3/2$, $x_2 = 5/2$, $x_3 = 0$

This is feasible & optimal for both LPs.

Different way:

Consider the LP: min $c^T x$, $Ax = b$, $x \geq 0$, $A \in \mathbb{R}^{m \times n}$

OPT is optimal value, x^* is optimal soln.

Want to get lower bound on objective.

Suppose we allow but penalize violations.

For $y \in \mathbb{R}^m$, $g(y) = \min_{x \geq 0} c^T x + y^T(b - Ax)$

For any y, $g(y) \leq c^T x^* + y^T(b - Ax^*) = c^T x^* = $ OPT

Say we maximize over all y: $\max_y g(y) = \max_y \min_{x \geq 0} y^T b + (c^T - y^T A)x$

(y are my violation penalties/prices)

What y makes sense? If $\exists j$ s.t. $c_j < y^T A_j$, then can choose $x_j \to \infty$ & $(c^T - y^T A)x \to -\infty$.

If $c_j \geq y^T A_j\ \forall j$, $x_j = 0$

Hence, $\max_y g(y) = \max y^T b$, $y^T A \leq c^T$ (or $A^T y \leq c$), and any soln for this LP is a lower bound on OPT.

Primal: min $c^T x$, $Ax = b$, $x \geq 0$

Dual: max $b^T y$, $A^T y \leq c$

Thus if x^* is optimal for primal, y^* for dual, then $c^T x^* \geq b^T y^*$

This is called weak duality.

Duality theory says that $c^T x^* = b^T y^*$ (will prove a bit later)

Similarly consider a (primal) LP in general form:

min $c^T x$

$Ax \geq b$, $A \in \mathbb{R}^{m \times n}$

lower bound: $g(y) = \min_x c^T x + y^T(b - Ax)$

Now for optimal soln x^*, could be that $a_j^T x^* > b_j$, if we allow $y_j < 0$, then $g(y)$ is no longer a lower bound.

Hence we restrict $y \geq 0$

Then for $y \geq 0$, $g(y) = \min_x c^T x + y^T(b - Ax)$

$\leq c^T x^* + y^T(b - Ax^*)$

$\leq c^T x^*$, since $b \leq Ax^*$

Maximizing our LB, $\max_{y \geq 0} g(y) = \max_{y \geq 0} \min_x y^T b + (c^T - y^T A)x$

What choice of y makes sense?

If $\exists i$: $c_i \neq y^T A_i$, choose x_i to be $+\infty / -\infty$

Hence, must choose $c^T = y^T A$

Then $\max_{y \geq 0} g(y) = \max y^T b$ s.t. $c^T = y^T A$, $y \geq 0$ or $\max b^T y$ s.t. $A^T y = c$, $y \geq 0$

Thus: Primal: min $c^T x$, $Ax \geq b$

Dual: max $b^T y$, $A^T y = c$, $y \geq 0$

More generally: min $\sum_j c_j x_j$ max $\sum_i b_i y_i$

Constraints: (except non neg)	variables
$a_i^T x = b_i$	free var y_i
$a_i^T x \geq b_i$	nonneg var y_i
variables:	constraints:
free variable x_j	$\sum_{\text{constraints } i} A_{ij} y_i = c_j$
nonneg var x_j	$\sum_{\text{constraints } i} A_{ij} y_i \leq c_j$

Eg.: min $x_1 + 2x_2 + 3x_3$

s.t. $-x_1 + 3x_2 = 5$

$2x_1 - x_2 + 3x_3 \geq 6$

$x_3 \leq 4$

$x_1 \geq 0$, $x_2 \leq 0$

Dual is max $5y_1 + 6y_2 - 4y_3$

$-y_1 + 2y_2 \leq 1$

$-3y_1 + y_2 \leq -2$

$3y_1 - y_3 = 3$

$y_2, y_3 \geq 0$, y_1 free

Theorem: Dual of the dual is the primal.

(proof skipped)

Theorem (Weak Duality): If x, y feasible for primal & dual respectively, then $c^T x \geq b^T y$

Proof (for LPs in standard form):

min $c^T x$, $Ax = b$, $x \geq 0$ max $b^T y$, $A^T y \leq c$

For any feasible x, y:

$c^T x = c^T x + y^T(b - Ax) = y^T b + (c^T - y^T A)x$

$\geq y^T b$

Corollary: (i) if optimal primal soln. is $-\infty$, dual must be infeasible

(ii) if optimal dual soln is ∞, primal must be infeasible

(iii) let x, y be feasible primal & dual solns. s.t. $c^T x = b^T y$. Then x, y are optimal.

Theorem: If an LP has a finite optimal cost, so does the dual, and the optimal costs are equal.

Proof: Assume LP in std. form: min $c^T x$, $Ax = b$, $x \geq 0$ | max $b^T y$, $A^T y \leq c$

has a finite optimal soln x^*.

Recall that simplex algo terminates w/ a basis $B \subseteq [n]$ (these are the non-zero vars.) & soln. x_B s.t.

$x_B = A_B^{-1} b$, $x_j = 0$ for $j \notin B$

& reduced costs are nonnegative:

$\forall x_j$, $c_j - c_B^T A_B^{-1} A_j \geq 0$

or $c^T - c_B^T A_B^{-1} A \geq 0$

Now choose $y^T = c_B^T A_B^{-1}$. Then $c^T - y^T A \geq 0$, so y is feasible.

& $c^T x = c_B^T x_B = c_B^T A_B^{-1} b = y^T b$ ∎

Primal		Dual
unbounded opt		infeasible
infeasible		unbounded opt
bounded opt	=	bounded opt
infeasible		infeasible

Complementary Slackness:

For weak duality

For any feasible x, y:

$c^T x = c^T x + y^T(b - Ax) = y^T b + (c^T - y^T A)x$

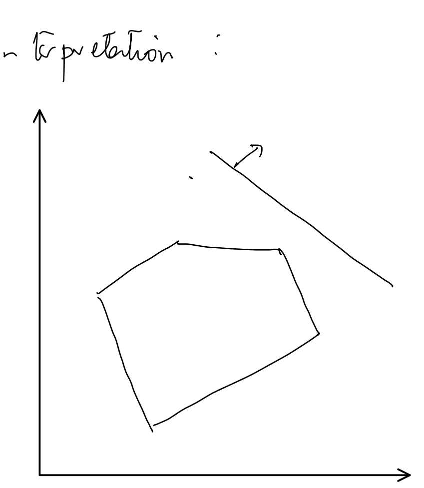
$\geq y^T b$

for equality, if $c_j > y^T A_j$, $x_j = 0$

Thus,

Theorem: x^*, y^* are optimal iff feasible & $\forall j \in [m]$, if $c_j > y^T A_j$, $x_j = 0$

Geometric interpretation:



max $c^T x$

$Ax \leq b$

min $b^T y$

$A^T y = c$

$y \geq 0$